

CONGRUENCES FOR RATIONAL POINTS ON VARIETIES OVER FINITE FIELDS

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ABSTRACT. We show that the number of rational points on the fibres of a proper morphism of smooth varieties over a finite field k whose generic fibre has a “trivial” Chow group of zero cycles is congruent to $1 \pmod{|k|}$. As a consequence we prove that there is a rational point on any degeneration of a smooth proper rationally chain connected variety over a finite field. We also obtain a generalisation of the Chevalley-Warning theorem.

1. INTRODUCTION

1.1. The main result of this paper is the following:

Theorem 1.1. *Let $f_i : X_i \rightarrow Y$, $i = 1, 2$ be proper dominant morphisms of smooth irreducible varieties over a finite field k and let $g : X_1 \rightarrow X_2$ be a dominant morphism over Y . Let Z_i be the generic fibre of f_i and assume that $g_* : CH_0(Z_{1\overline{k(X_1)}})_{\mathbb{Q}} \rightarrow CH_0(Z_{2\overline{k(X_1)}})_{\mathbb{Q}}$ is an isomorphism. Then for all $y \in Y(k)$,*

$$|f_1^{-1}(y)(k)| \equiv |f_2^{-1}(y)(k)| \pmod{|k|}.$$

Specialising to the case $X_2 = Y$ and $f_2 = Id_Y$, we obtain:

Corollary 1.2. *Let $f : X \rightarrow Y$ be a proper dominant morphism of smooth irreducible varieties over a finite field k . Let Z be the generic fibre of f and assume that $CH_0(Z_{\overline{k(X)}})_{\mathbb{Q}} = \mathbb{Q}$. Then for all $y \in Y(k)$, $|f^{-1}(y)(k)| \equiv 1 \pmod{|k|}$.*

When $Y = Spec(k)$, the above corollary reduces to a theorem of H. Esnault [10]. Since we do not assume that f is smooth, we are able to obtain congruences even for singular varieties.

An immediate consequence of Corollary 1.2 is that for $f : X \rightarrow Y$ as above, $|X(k)| \equiv |Y(k)| \pmod{|k|}$. By taking Y to be a point in Theorem 1.1 we obtain:

Corollary 1.3. *Let $g : X_1 \rightarrow X_2$ be a dominant morphism of smooth proper varieties over a finite field k . If $g_* : CH_0(X_{\overline{k(X)}})_{\mathbb{Q}} \rightarrow CH_0(Y_{\overline{k(X)}})_{\mathbb{Q}}$ is an isomorphism, then $|X(k)| \equiv |Y(k)| \pmod{|k|}$.¹*

Simple examples show that in contrast to Corollary 1.2, the assumption that X_1 and X_2 be proper cannot be omitted here.

The triviality of the Chow group of zero cycles of degree 0, or even rational chain connectedness² is not sufficient to guarantee the existence of a rational point for proper varieties

¹This can also be proved using the method of [13].

²We learnt of such an example, due to J. Kollar, from a talk by J. Iyer at the University of Essen in May 2003; but see Remark 3.4

over finite fields which are not smooth. However, using alterations, we give a criterion for the existence of rational points which can be applied to all degenerations of smooth rationally chain connected varieties.

Corollary 1.4. *Let $f : X \rightarrow Y$ be a proper dominant morphism of irreducible varieties over a finite field k with Y smooth. Let Z be the generic fibre of f and assume that Z is smooth and $CH_0(Z_{\overline{k(X)}})_{\mathbb{Q}} = \mathbb{Q}$. Then for any $y \in Y(k)$, $f^{-1}(y)(k) \neq \emptyset$.*

The corollary below generalises the Chevalley–Warning theorem [12], which is the special case $P = \mathbb{P}^n$ and $L_i = \mathcal{O}(d_i)$ with $\sum d_i \leq n$.

Corollary 1.5. *Let P be a smooth projective geometrically irreducible variety over a finite field k . Let L_1, \dots, L_r be very ample line bundles on P such that $(K_P \otimes L_1 \otimes \dots \otimes L_r)^{-1}$ is ample, where K_P is the canonical bundle of P . For $i = 1, 2, \dots, r$, let $s_i \in H^0(P, L_i)$. Then*

$$|\{x \in P(k) \mid s_i(x) = 0, i = 1, 2, \dots, r\}| \equiv 1 \pmod{|k|}.$$

Note that the congruence formula of Katz [6], when it applies, only gives congruences modulo $p = \text{char}(k)$. It would be interesting to know whether an analogue of the Ax–Katz theorem [14] holds in the above situation or whether all low degree intersections as above over C_1 fields always have rational points.

Our proofs, as is Esnault’s, are based on the method of decomposing the diagonal originated by Bloch, the Grothendieck–Lefschetz trace formula and Deligne’s integrality theorem. The novelty of our approach lies in the use of proper correspondences which allows us to deal with non-proper Y ; this is crucial for obtaining congruences for (or even just existence of) rational points on singular varieties. The main technical ingredient for this is the refined intersection theory of Fulton and MacPherson [11].

2. PROPER CORRESPONDENCES

2.1. In this section we introduce the concept of proper correspondences and prove some of its properties.³ Most of the proofs are simple modifications of those in Fulton’s book [11] and we prove only what we need for later use; several other results in [11, Chapter 16] for (usual) correspondences have analogues for proper correspondences.

Definition 2.1. Let X and Y be smooth irreducible varieties over a field k . The group of *proper correspondences* from X to Y , $PCorr(X, Y)$, is the free abelian group generated by irreducible subvarieties $\Gamma \subset X \times Y$ which are proper over Y modulo the subgroup generated by elements of the form

$$\{div(f) \mid f \in k(Z)^*, Z \subset X \times Y \text{ irreducible and proper over } Y\}.$$

This is clearly a graded abelian group; we shall grade it by dimension (lower indices) or codimension (upper indices) as is convenient. For a cycle Γ (not necessarily irreducible) in the free abelian group as above we shall denote its class in $PCorr(X, Y)$ by $[\Gamma]$.

³A. Nair has informed us that a variant of this definition has been considered, in a different context, by E. Urban in the preprint: Sur les représentations p -adiques associées aux représentations cuspidales de GSp_4/\mathbb{Q} , 2001.

If $f : X \rightarrow Y$ is a proper morphism, then the graph of f gives an element $[\Gamma_f]$ of $PCorr(X, Y)$. We shall show below that proper correspondences induce maps on cohomology generalising the maps induced by proper morphisms.

2.2. We recall some properties of étale (co)homology and cycle class maps that we shall need. The main reference is [7, Exposés VI-IX]; we note that the quasi-projectivity hypotheses there can be removed using the methods of [11]. The compatibility of refined intersection products and refined cycle class maps stated below can be deduced using the methods of [11, Chapter 19]; see also [3].

Let K be an algebraically closed field and fix a prime number $l \neq \text{char}(K)$. For a variety X over K let $H^i(X) := H_{et}^i(X, \mathbb{Q}_l)$, $H_c^i(X) := H_{c, et}^i(X, \mathbb{Q}_l)$ and let $H_Z^i(X) := H_{Z, et}^i(X, \mathbb{Q}_l)$ for Z a closed subvariety of X . We also let $H_i(X)$ denote the locally finite (or Borel-Moore) homology $H_i^{et}(X, \mathbb{Q}_l)$. For any of the above groups H , we denote by $H(n)$, for an integer n , the corresponding Tate twisted group.

For a cycle $\alpha = \sum_i a_i Z_i$, on a variety X , we denote by $|\alpha|$ the support $\cup_i Z_i$ of α . The following properties are proved in the references cited above:

- (1) (Projection formula) For any variety X there are cap product maps $H^i(X) \otimes H_j(X) \rightarrow H_{j-i}(X)$ such that if $f : X \rightarrow Y$ is a proper morphism, $u \in H^i(Y), v \in H_j(X)$ then $f_*(f^*(u) \cap v) = u \cap f_*(v)$ in $H_{j-i}(Y)$.
- (2) For Z an irreducible variety of dimension n , $H_{2n}(Z)(-n)$ is one dimensional.
- (3) For α a cycle of dimension k on a variety X the fundamental class defines a canonical element $cl(\alpha) \in H_{2k}(|\alpha|)(-k)$. This maps to an element, also denoted by $cl(\alpha)$, in $H_{2k}(X)(-k)$ which is zero if $\alpha = 0 \in CH_k(X)$.
- (4) For X smooth of pure dimension n there is a canonical isomorphism $H_{2k}(|\alpha|)(-k) \cong H_{|\alpha|}^{2n-2k}(X)(n-k)$, so we also get an element $cl(\alpha) \in H_{|\alpha|}^{2n-2k}(X)(n-k)$.
- (5) For $f : X \rightarrow Y$ a morphism and α a cycle on X of dimension n whose support is proper over Y , $f_*(cl(\alpha)) = cl(f_*(\alpha))$ in $H_{f(|\alpha|)}^*(Y)(-n)$.
- (6) For α and β cycles of pure dimension k and l respectively in a smooth irreducible variety X of dimension n

$$cl(\alpha \cdot \beta) = cl(\alpha) \cup cl(\beta) \in H_{|\alpha| \cap |\beta|}^{4n-2(k+l)}(X)(n-k-l) ,$$

where the product on the left is the refined intersection product.

2.3. Let $\dim(X) = n$, $\dim(Y) = m$ and let Γ be a representative for an element of $PCorr_r(X, Y)$. Then Γ induces a linear map $\Gamma_* : H^i(X) \rightarrow H^{2m-2r+i}(Y)(m-r)$ as the composite of the sequence of linear maps:

$$\begin{aligned} H^i(X) &\xrightarrow{p_X^*} H^i(X \times Y) \xrightarrow{\cdot cl(\Gamma)} H_{|\Gamma|}^{i+2(n+m-r)}(X \times Y) \cong \\ &H_{2r-i}(|\Gamma|)(-r) \xrightarrow{p_{Y*}} H_{2r-i}(Y)(-r) \cong H^{2m-2r+i}(Y)(m-r) . \end{aligned}$$

Since X and Y are smooth, by duality we also get maps $\Gamma^* : H_c^i(Y) \rightarrow H_c^{2n-2r+i}(X)(n-r)$.

Now suppose that k is a finite field and K an algebraic closure of k . All the cohomology groups discussed above, for varieties over K which are base changed from varieties over k , have a continuous action of $\text{Gal}(K/k)$. The maps Γ^* and Γ_* are then compatible with the Galois action.

The following lemma implies that the maps Γ_* and Γ^* only depends on $[\Gamma] \in PCorr(X, Y)$.

Lemma 2.2. *Let Γ be as above. Suppose there exists a closed subvariety $Z \subset X \times Y$ such that $|\Gamma| \subset Z$, $[\Gamma] = 0$ in $CH_*(Z)$ and $p_Y|_Z : Z \rightarrow Y$ is proper. Then Γ^* and Γ_* are the zero maps.*

Proof. Suppose Z is a closed subvariety of $X \times Y$ such that the projection $p_Y|_Z : Z \rightarrow Y$ is proper. It follows from the projection formula that Γ_* can also be defined as,

$$H^i(X) \xrightarrow{p_X|_Z^*} H^i(Z) \xrightarrow{\cdot cl(\Gamma)} H_{2r-i}(Z)(-r) \xrightarrow{p_Y}_* H_{2r-i}(Y)(-r) \cong H^{2m-2r+i}(Y)(m-r),$$

where $cl(\Gamma)$ is considered as an element in $H_{2r}(Z)(-r)$. The lemma follows from the fact that $cl(\Gamma) = 0$ in $H_{2r}(Z)(-r)$. The statement for Γ^* follows by duality. \square

2.4. Let X, Y, Z be smooth irreducible varieties over K and let $[\Gamma_1]$ (resp. $[\Gamma_2]$) be a proper correspondence from X to Y (resp. Y to Z). Analogous to the definition of composition of correspondences [11, Chapter 16], we define $[\Gamma_2] \circ [\Gamma_1] \in PCorr(X, Z)$ by

$$[\Gamma_2] \circ [\Gamma_1] = [p_{XZ}_*(p_{XY}^*(\Gamma_1) \cdot p_{YZ}^*(\Gamma_2))],$$

where the p 's denote the projection morphisms from $X \times Y \times Z$ and the product is the refined intersection product. Note that $p_{XY}^*(\Gamma_1) \cdot p_{YZ}^*(\Gamma_2)$ is a cycle which is well defined upto rational equivalence on $|\Gamma_1| \times_Y |\Gamma_2|$. Since $|\Gamma_1| \times_Y |\Gamma_2|$ is proper over Z , its image in $X \times Z$ is also proper over Z , so $[\Gamma_2] \circ [\Gamma_1]$ is a well defined element of $PCorr(X, Z)$.

Lemma 2.3. *Let X, Y, Z and Γ_1, Γ_2 be as above. Then $(\Gamma_2 \circ \Gamma_1)_* = (\Gamma_2)_* \circ (\Gamma_1)_*$ as maps from $H^*(X)$ to $H^*(Z)$ and $(\Gamma_2 \circ \Gamma_1)^* = (\Gamma_1)^* \circ (\Gamma_2)^*$ as maps from $H_c^*(Z)$ to $H_c^*(X)$.*

Proof. The key point is the compatibility of the refined cycle class maps with refined intersection products.

Let $a \in H^*(X)$. Then

$$\begin{aligned} (\Gamma_2)_* \circ (\Gamma_1)_*(a) &= p_Z^{YZ}_*(cl(\Gamma_2) \cdot p_Y^{YZ*}(p_Y^{XY*}(cl(\Gamma_1) \cdot p_X^{XY*}(a)))) \\ &= p_Z^{YZ}_*(cl(\Gamma_2) \cdot p_{YZ}^{XYZ*}(p_{XY}^{XYZ*}(cl(\Gamma_1) \cdot p_X^{XY*}(a)))) \\ &= p_Z^{YZ}_*(p_{YZ}^{XYZ*}(p_{YZ}^{XYZ*}(cl(\Gamma_2)) \cdot p_{XY}^{XYZ*}(cl(\Gamma_1) \cdot p_X^{XY*}(a)))) \\ &= p_Z^{YZ}_*(p_{YZ}^{XYZ*}(p_{YZ}^{XYZ*}(cl(\Gamma_2)) \cdot p_{XY}^{XYZ*}(cl(\Gamma_1)) \cdot p_X^{XYZ*}(a))) \\ &= p_Z^{XZ}_*(p_{XZ}^{XYZ*}(p_{YZ}^{XYZ*}(cl(\Gamma_2)) \cdot p_{XY}^{XYZ*}(cl(\Gamma_1)) \cdot p_X^{XYZ*}(a))) \\ &= p_Z^{XZ}_*(cl(\Gamma_2 \circ \Gamma_1) \cdot p_X^{XZ*}(a)) \\ &= (\Gamma_2 \circ \Gamma_1)_*(a) \end{aligned}$$

We use the projection formula several times along with compatibility of the cycle class map with smooth pullbacks, products and proper pushforwards. \square

2.5. The key technical result which allows us to deduce congruences from cycle theoretic information is the following:

Proposition 2.4. *Let X, Y, Γ be as above and assume that $m = n = r$. If $\dim(p_X(|\Gamma|)) < n$ then all the eigenvalues of (the geometric) Frobenius acting on $\Gamma^*(H_c^i(Y)) \subset H_c^i(X)$ are algebraic integers which are divisible by $|k|$.*

Proof. Replacing k by a finite extension does not affect the conclusion of the lemma, so without loss of generality we may assume that Γ is a geometrically irreducible subvariety of

$X \times Y$. Using the definition of Γ_* as given in the proof of Lemma 2.2, Γ_* is the composite of the following maps:

$$H^i(X) \xrightarrow{p_X^*} H^i(\Gamma) \longrightarrow H_{2n-i}(\Gamma)(-n) \xrightarrow{p_Y|\Gamma_*} H_{2n-i}(Y)(-n) \cong H^i(Y),$$

where we have used the hypothesis that $m = n = r$.

Suppose $\pi : \Gamma' \rightarrow \Gamma$ is a proper dominant generically finite morphism with Γ' smooth and geometrically irreducible. Then the projection formula shows that we may replace Γ with Γ' and p_X (resp. p_Y) with $p_X\pi$ (resp. $p_Y\pi$) in the above without changing the image of the composite.

Let W be the Zariski closure of $p_X(\Gamma)$ in X and let $\dim(W) = t$. By the theorem of De Jong [4, 4.1] we may find Γ' as above, $\sigma : W' \rightarrow W$ with σ proper dominant generically finite and W' smooth, and $p : \Gamma' \rightarrow W'$ such that $p_X\pi = i_W\sigma p$, where $i_W : W \rightarrow X$ is the inclusion. Since $p_X\pi = (i_W\sigma)p$ and both Γ' and W' are smooth, it follows from the functoriality of pushforward maps that $(p_X\pi)_* = (i_W\sigma)_*p_*$. By the remarks of the previous paragraph we see that the image of Γ_* is the same as that of the composite of the sequence:

$$H^i(X) \longrightarrow H^i(W') \longrightarrow H^i(\Gamma') \longrightarrow H^i(Y).$$

Since Γ' and W' are smooth, by duality we get a factorisation of Γ^* as a composite of maps,

$$H_c^j(Y) \longrightarrow H_c^j(\Gamma') \longrightarrow H_c^{2(t-n)+j}(W')(t-n) \longrightarrow H_c^j(X).$$

By Deligne's integrality theorem [6], Exposé XXI, Corollaire 5.5.3, all the eigenvalues of Frobenius on $H_c^*(W')$ are algebraic integers. Since $n - t > 0$ and the geometric Frobenius acts on $\mathbb{Q}_l(t-n)$ by $|k|^{n-t}$, the proposition follows. \square

3. PROOFS OF THE MAIN RESULTS

Using the results of the previous section, we now give the proofs of the results stated in the introduction.

3.1.

Proof of Theorem 1.1. If Y is not geometrically irreducible, then $Y(k) = \emptyset$ so there is nothing to prove. The hypothesis on the Chow group implies that the geometric connected components of Z_1 and Z_2 are in bijective correspondence. Since X_1 and X_2 are smooth, we can assume that X_1 , X_2 and Y are geometrically irreducible.

Let W be an irreducible subvariety of X_1 which maps generically finitely and dominantly to X_2 and let d be the degree of W over X_1 . Let Γ_g be the graph of g in $X_1 \times_k X_2$, let Γ_W be the transpose of the graph of $g|_W$ embedded in $X_2 \times_k X_1$ and let $\Gamma_1 = (\Gamma_W \circ \Gamma_g)/d$. Since W is a subvariety of X_1 , Γ_W is a proper correspondence, and Γ_g is a proper correspondence since g is proper. Hence Γ_1 is a proper correspondence of dimension $n_1 = \dim X_1$.

Let V_2 be the open subset of X_2 over which $g|_W$ is finite and let $V_1 = g^{-1}(V_2)$. By the construction of W , $p_{12}^*(\Gamma_g)$ and $p_{23}^*(\Gamma_W)$, which are subvarieties of $X_1 \times_k X_2 \times_k X_1$, meet properly when pulled back to $V_1 \times_k X_2 \times_k X_1$. It follows that one can write $\Gamma_1 = \Gamma'_1 + \Gamma''_1$ in $PCorr(X_1, X_1)$ where $(Id_{X_1} \times g)_*(\Gamma'_1) = \Gamma_g$ in $PCorr(X_1, X_2)$ and $p_1(|\Gamma''_1|)$ is a proper subvariety of X_1 .

Let Δ_{X_1} be the diagonal in $X_1 \times_k X_1$ and consider the proper correspondence $\Gamma_2 := \Delta_{X_1} - \Gamma_1$ and its restriction (i.e. flat pullback) γ_2 to $Z_1 \times_{k(Y)} Z_1$. Since $(Id_{X_1} \times g)_*(\Delta_{X_1}) = \Gamma_g$, it follows from the previous paragraph that $(Id_{Z_1} \times g|_{Z_1})_*(\gamma_2)$ can be represented by a cycle on $Z_1 \times_{k(Y)} Z_2$ which becomes zero when restricted to $k(Z_1) \times_{k(Y)} Z_2$. Since the map

$g_* : CH_0(Z_{1\overline{k(X_1)}})_{\mathbb{Q}} \rightarrow CH_0(Z_{2\overline{k(X_1)}})_{\mathbb{Q}}$ is injective, it follows that γ_2 can be represented by a cycle on $Z_1 \times_{k(Y)} Z_1$ whose support maps to a proper subvariety of Z_1 by the projection to the first factor. By taking the Zariski closures in $X_1 \times_Y X_2$, it follows that Γ_2 can be represented by a proper correspondence on $X_1 \times_k X_1$ whose support maps to a proper subvariety of X_1 by the projection to the first factor.

By Proposition 2.4 all the eigenvalues of Frobenius acting on the image of Γ_2^* in $H_c^*(X_1)$ are divisible by $|k|$. It follows from Lemma 2.3 and the definition of Γ_1 , that the image of Γ_1^* is contained in the image of g^* . By the definition of Γ_2 , we conclude that all the eigenvalues of Frobenius acting on the cokernel of $g^* : H_c^*(X_2) \rightarrow H_c^*(X_1)$ are divisible by $|k|$.

The hypotheses of the theorem, and the above discussion, are not affected if we replace Y by an open subvariety U and X_i by $f_i^{-1}(U)$, $i = 1, 2$, so we may assume that Y has only one rational point y . Applying the Grothendieck-Lefschetz trace formula and the statement on eigenvalues above, we conclude that the $|X_1(k)| \equiv |X_2(k)| \pmod{|k|}$. Each rational point of X_i must lie over the unique rational point y of Y , hence the theorem follows. \square

Corollary 3.1. *Let $f : X \rightarrow Y$ be a proper dominant generically finite morphism of smooth irreducible varieties over a finite field k such that the extension of function fields $k(Y) \rightarrow k(X)$ is purely inseparable. Then for any $y \in Y(k)$, $|f^{-1}(y)(k)| \equiv 1 \pmod{|k|}$.*

Proof. Since $(Z_{\overline{k(X)}})_{red}$ is isomorphic to $Spec(\overline{k(X)})$, the hypothesis on CH_0 is trivially satisfied. \square

Remark 3.2. Since we only use intersection theory (resp. cohomology groups) with rational (resp. \mathbb{Q}_l) coefficients, the above results hold even when X and Y are quotients of smooth varieties by finite groups.

3.2.

Proof of Corollary 1.4. As in the proof of Theorem 1.2, we may assume that X and Y are geometrically irreducible and that Y has a unique rational point.

By a result of De Jong [5, 5.15] there exists an irreducible variety X' over k which is the quotient of a smooth variety by a finite group and a proper dominant generically finite morphism $\pi : X' \rightarrow X$ such that the extension of function fields $k(X) \rightarrow k(X')$ is purely inseparable. Let $f' = f\pi : X' \rightarrow Y$ and let Z' be the generic fibre of f' . Since X' is irreducible over k , Z' is irreducible over $k(Y)$. Z is geometrically irreducible since it is smooth over $k(Y)$; since the extension of function fields induced by the map $Z' \rightarrow Z$ is $k(X) \rightarrow k(X')$, Z' is also geometrically irreducible. The induced morphism $(Z'_{\overline{k(X')}})_{red} \rightarrow Z_{\overline{k(X)}}$ thus satisfies the assumptions of Lemma 3.5 below, so $CH_0(Z'_{\overline{k(X')}})_{\mathbb{Q}} = \mathbb{Q}$. Applying Corollary 1.2 (cf. Remark 3.2) to the morphism $f' : X' \rightarrow Y$ we see that $X'(k) \neq \emptyset$. Thus $X(k) \neq \emptyset$. \square

Remark 3.3. For singular X it is not always true that $|f^{-1}(y)(k)| \equiv 1 \pmod{|k|}$, however the only examples we know where this fails are non-normal.

Remark 3.4. $X(k) \neq \emptyset$ if X is a proper variety over a finite field k which is rationally connected i.e. any two general points of $X(\Omega)$ are contained in an irreducible rational curve in X defined over Ω , where $\Omega \supset k$ is a universal domain. (To prove this we may replace X by X' as in the proof of Corollary 1.4. Since the extension of function fields $k(X) \rightarrow k(X')$ is purely inseparable, it follows that X' is also rationally connected, so $CH_0(X'_{\overline{k(X')}})_{\mathbb{Q}} = \mathbb{Q}$. By Remark 3.2 it follows that $X'(k) \equiv 1 \pmod{|k|}$, hence $X'(k) \neq \emptyset$. Thus $X(k) \neq \emptyset$.)

However, a degeneration of a rationally connected variety is in general only rationally chain connected, so one cannot use this to prove Corollary 1.4 even if Z is rationally connected.

Lemma 3.5. *Let $f : X \rightarrow Y$ be a proper dominant morphism of irreducible varieties over an algebraically closed field K . Assume that Y is smooth, f is generically finite and the extension of function fields $K(Y) \rightarrow K(X)$ is purely inseparable. Then $f_* : CH_0(X)_{\mathbb{Q}} \rightarrow CH_0(Y)_{\mathbb{Q}}$ is an isomorphism.*

Proof. f_* is surjective because f is surjective. Using the refined intersection theory of [11] and the hypothesis on Y , we see that there is a natural map $f^* : CH_0(Y)_{\mathbb{Q}} \rightarrow CH_0(X)_{\mathbb{Q}}$. By the assumptions on f there exists an open subset U of Y such that for $V = f^{-1}(U)$, $f|_V : V \rightarrow U$ is a bijection. By the moving lemma (which is trivial for zero cycles) $CH_0(X)$ (resp. $CH_0(Y)$) is generated by the closed points in V (resp. U). This shows that f^* is a surjection since for $y \in U$, $f^*([y]) = [f^{-1}(y)]$. Since f_*f^* is multiplication by $\deg(f)$, it follows that f_* is an isomorphism. \square

3.3.

Proof of Corollary 1.2. Let $Y = \prod_i H^0(P, L_i)$ and let $X \subset Y \times P$ be the zero scheme of the universal section of $\bigoplus_i p_P^* L_i$. Since the L_i are basepoint free, the map $X \rightarrow P$ is smooth, hence X is also smooth. Since the L_i are assumed to be very ample, Bertini's theorem implies that the generic fibre Z of the projection $f : X \rightarrow Y$ is smooth. The assumptions on the L_i and the adjunction formula imply that Z is a Fano variety. By a theorem of Campana [2] and Kollar-Miyaoka-Mori [15] Z is rationally chain connected, so $CH_0(Z_{\overline{k(X)}})_{\mathbb{Q}} = \mathbb{Q}$. The proof is concluded by applying Corollary 1.2 to f . \square

Remark 3.6. The proof shows that instead of assuming that the L_i are very ample it suffices to assume that they are basepoint free and that Z is smooth.

A motivic version of the above corollary in the case of hypersurfaces in \mathbb{P}^n is proved in [1].

4. FURTHER QUESTIONS

It seems likely that the following mixed characteristic analogue of Corollary 1.4 has a positive answer:

Question 4.1. let K be a finite extension of \mathbb{Q}_p , \mathcal{O} its ring of integers, k the residue field and X a smooth proper variety over K such that $CH_0(X_{\overline{K(X)}})_{\mathbb{Q}} = \mathbb{Q}$. If $\mathcal{X} \rightarrow \text{Spec}(\mathcal{O})$ is a proper scheme with generic fibre isomorphic to X and closed fibre X_0 , then is $X_0(k) \neq \emptyset$?

From our proofs we do not obtain any information about the valuations of the eigenvalues of Frobenius acting on the étale cohomology of the fibres of f . One is thus lead to ask the following:

Question 4.2. Let $f : X \rightarrow Y$ be a proper dominant morphism of smooth irreducible varieties over a finite field k . Let Z be the generic fibre of f and assume that $CH_0(Z_{\overline{k(X)}})_{\mathbb{Q}} = \mathbb{Q}$. Then for all $y \in Y(k)$ and $i > 0$, does $|k|$ divide all the eigenvalues of Frobenius acting on $H^i((f^{-1}(y))_{\overline{k}}, \mathbb{Q}_l)$ ($l \neq \text{char}(k)$)?

Theorem 1.1 has the following Hodge theoretic analogue:

Theorem 4.3. *Let $f_i : X_i \rightarrow Y$, $i = 1, 2$, be proper dominant morphisms of smooth irreducible varieties over \mathbb{C} and let $g : X_1 \rightarrow X_2$ be a dominant morphism over Y . Let Z_i be the generic fibre of f_i and assume that $g_* : CH_0(Z_{1\overline{\mathbb{C}(X_1)}})_{\mathbb{Q}} \rightarrow CH_0(Z_{2\overline{\mathbb{C}(X_1)}})_{\mathbb{Q}}$ is an isomorphism. Then $gr_0^F(Coker(g^* : H_c^i(X_1, \mathbb{C}) \rightarrow H_c^i(X_2, \mathbb{C}))) = 0$ for all i , where F^\bullet denotes the Hodge filtration of Deligne.*

The proof is essentially the same as that of Theorem 1.1 so we omit the details: one only needs to replace Lemma 2.4 by its Hodge theoretic counterpart.

The Hodge version of Question 4.2 is:

Question 4.4. *Let $f : X \rightarrow Y$ be a proper dominant morphism of smooth irreducible varieties over \mathbb{C} . Let Z be the generic fibre of f and assume that $CH_0(Z_{\overline{\mathbb{C}(X)}})_{\mathbb{Q}} = \mathbb{Q}$. Then for all $y \in Y(\mathbb{C})$ and $i > 0$, is $gr_0^F(H^i(f^{-1}(y), \mathbb{C})) = 0$?*

It seems likely, as was suggested to us by P. Brosnan, that there should be a purely motivic statement from which Theorems 1.2 and 4.3 should follow after taking realisations. The category of motives over a base of Corti and Hanamura [3] would seem to be a natural choice in which to formulate such a statement.

Remark 4.5. After this paper was submitted both Question 4.2 and (a stronger form of) Question 4.4 were shown to have positive answers by H. Esnault; her results appear in the appendix to this article [8]. A positive answer to Question 4.1 in case X is regular was also obtained by her in [9]; the results of that paper include a stronger version of Corollary 1.2 in the case that Y is a curve.

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